Locating Complex Roots in the Graphs of Rational Functions

Michael J. Bossé
e-mail: bossemj@appstate.edu
Appalachian State University
Boone, NC, USA

William Bauldry
e-mail: bauldrywc@appstate.edu
Appalachian State University
Boone, NC, USA

Steven H. Otey
e-mail: oteysh@appstate.edu
Appalachian State University
Boone, NC, USA

Abstract

The paper also draws on experiences gained in working with pilot schools on the introduction of the software to teachers whose own background in 3D (and other) geometry is not very strong. In this paper we address the question: Given solely the graph of a rational function, from which both the numerator and the denominator are real monic polynomials, can the approximate location of the complex roots from either the numerator or denominator be determined? This is an extension of the authors' previous work on locating complex roots of polynomial functions. This paper demonstrates that, under a set of simple conditions, the locations of these complex roots can be approximated.

1. Introduction

Through a number of investigations, we have considered visualizing the location of complex roots based on the graphs of quadratic, cubic, and quartic equations [2, 3, 5] and quintic real polynomials [4] and novel findings that evolved from these investigations. As a natural extension, we now consider the graphs of rational functions and attempt to locate complex roots based solely on the graphs of these functions. The following develops some observations and notions discovered through this investigation. It is hoped that this investigation will further motivate student interest in the graphs of functions and their algebraic connections.

A preliminary form of this paper was used as instructional materials in undergraduate and graduate real analysis classes at Appalachian State University in Boone, NC, USA. Students reported to have expeditiously come to a deeper understanding of the topic of rational functions, graphs, real and complex zeros through the prose supported by the dynamic graphing applets. Additionally, this material was used in a number of professional development scenarios for high school teachers. Many expressed a desire to use the materials in their own upper level high school classes, believing that much of the material was accessible to their students. Thus, the intended audience of this paper includes upper grades high school students, high school teachers, undergraduate and graduate mathematics majors, and university faculty.

2. The role and power of conjectures

The following discussions take the form of theorems leading to further observations and conjectures. While the development and proof of theorems is central to mathematics, one cannot devalue the role of observations which lead to conjectures and, if proven, then to theorems. The question may naturally arise, "What leads to observation and conjecture?". The answer is quite simple: inquiry and imagination. Bailey and Borwein [1] discuss the value of experimental mathematics, where students are allowed to interact with mathematical ideas – often in a dynamic computer mathematical system – and investigate, make observations, and generalize through conjectures. Through these experimental opportunities, students become excited about the mathematics and enthusiastically take hold of findings as their own discoveries.

More than twenty-five years ago, one of the authors asked the question: Given only the graph of a real polynomial function, can one locate the complex roots when they exist? He briefly partnered with a colleague to investigate the question, only to be told that there was no solution. Undaunted, through the decades, the author revisited this question for moments at a time, only to be put away for years before being briefly reinvestigated again and then again placed on the shelf. This cycle persisted until one day the author decided that he would partner with another researcher and persist to a solution. As a consequence, there indeed were solutions [2, 3, 4, 5]. However, these solutions could never have been found without imagination to frame the question and inquiry to pursue the solution.

The question now at hand is: Given only the graph of a rational function with both numerator and denominator as real polynomial functions, can one locate the complex roots in either the numerator or the denominator when they exist? At first blush, the answer might again seem to be in the negative. However, employing ideas previously developed regarding complex roots of polynomials, observations can be made. We invite the reader to imagine and inquire with us through some conjectures. Provided in this article are dynamic apples to allow the reader to experiment with rational functions. We also invite others to prove some of the conjectures that follow.

3. Mathematical Background

All polynomial functions investigated in this paper are real polynomials (i.e., polynomials with real coefficients). Rational functions are herein defined as $q(x) = n(x)/d(x)$, where $n(x)$ and $d(x)$ are real polynomials.

Authors [2, 3, 4, 5] found that, for polynomials with complex roots $(a, \pm bi)$, the graphical behavior of the functions having complex roots is only observable when b is sufficiently close to the x-axis. In summary: when b is small (sufficiently close to 0), the graph of the polynomial will produce extrema pointing toward the x-axis in the neighborhood of a; when b is of medium magnitude, the graph will demonstrate a noticeable "flattening" in the neighborhood of a; and when b is large (departs significantly from the x-axis), the behavior of the graph cannot be observed to determine the location of complex roots. Employing more formal mathematical language, the flattening of the graph can be stated as diminishing the absolute value of the concavity of the graph in the region in question. (We consider this in more detail below.) These lead to the following conditions necessary for visualizing the location of complex roots on rational functions.

> **Conditions:** For all complex roots $(a, \pm bi)$ in either the numerator or denominator, b is assumed sufficiently close to the x-axis in order to observe the stated behavior. In the instance where $q(x) = n(x)/d(x)$, where $n(x)$ and $d(x)$ are nonzero monic polynomials, the complex roots of $n(x)$ and $d(x)$ will be denoted $\left(a_{n_j}, \pm b_{n_j}\right)$ and $\left(a_{d_k}, \pm b_{d_k}\right)$ respectively. Real roots are denoted r_{n_j} and r_{d_k} respectively.

4. Rational function theorems

The findings described in this paper are discussed for the two cases: where complex roots are stacked – or nearly so – above real or complex roots, or when all roots are sufficiently horizontally separated such that none of the roots act as if they are stacked roots. We now consider a number of theorems regarding these cases. These theorems are instrumental in understanding the later observations made of the behavior of graphs of rational functions.

4.1 Horizontally Separated (Unstacked) Roots

We first consider the case where complex roots in the numerator or denominator are sufficiently horizontally distinct so that they do not act like roots vertically stacked over any other roots, either from the numerator or the denominator.

Theorem 1. As $b_i \to 0$ and $f'(x) = 0$, then

$$f(x)=\frac{(x-r_1)\left((x-a_1)^2+b_1^2\right)}{(x-r_2)\left((x-a_2)^2+b_2^2\right)} \quad \text{Þ} \quad f(x)=\frac{(x-r_1)(x-a_1)^2}{(x-r_2)\left((x-a_2)^2+b_2^2\right)} \quad \text{or} \quad f(x)=\frac{(x-r_1)\left((x-a_1)^2+b_1^2\right)}{(x-r_2)\left((x-a_2)^2\right)}$$

$$f(x)=\frac{\left((x-a_1)^2+b_1^2\right)\left((x-a_2)^2+b_2^2\right)}{\left((x-a_3)^2+b_3^2\right)\left((x-a_4)^2+b_4^2\right)} \quad \text{Þ} \quad f(x)=\frac{(x-a_1)^2\left((x-a_2)^2+b_2^2\right)}{\left((x-a_3)^2+b_3^2\right)\left((x-a_4)^2+b_4^2\right)} \quad \text{or} \quad f(x)=\frac{\left((x-a_1)^2+b_1^2\right)\left((x-a_2)^2+b_2^2\right)}{(x-a_3)^2\left((x-a_4)^2+b_4^2\right)}$$

Thus, when $b_i \to 0$, $f'(x)$ will have an extremum at a_i. To demonstrate this, assume that r_1 , r_2, a_1, and a_2 are "sufficiently separated." Define

$$q : x \mapsto \frac{(x-r_1)\left((x-a_1)^2+b_1{}^2\right)}{(x-r_2)\left((x-a_2)^2+b_2{}^2\right)}$$

Then

$$q'(a_1) = -2b_1{}^2 \cdot \frac{a_1{}^3 - \frac{1}{2}(2a_2+3r_1+r_2)a_1{}^2 + 2r_1\left(a_2+\frac{1}{2}r_2\right)a_1 - \frac{1}{2}\left[(r_1-r_2)\left(a_2{}^2+b_2{}^2\right)+2a_2r_1r_2\right]}{(a_1-r_2)^2\left((a_1-a_2)^2+b_2{}^2\right)^2}$$

If $b_1 = 0$, then $x = a_1$ is a double real root and an extremum of q. As b_1 increases, leading to the scaling of $q'(a_1) = \mathcal{O}(b_1^2)$, we see that q moves off the axis, retaining the extrema until b_1 becomes sufficiently large. Since polynomials are continuous functions of their coefficients, and b_1 only appears in the numerator polynomial, the extremum remains 'near' the complex root (a_1, b_1) until b_1 becomes large enough for the extrema to become 'flattened'; i.e., $q''(a_1)$ goes toward 0 as b_1 becomes large. Since the extrema being 'lifted' is a double root, it must also 'point towards' the x-axis.

If $b_2 = 0$, then $x = a_2$ is a double pole or vertical asymptote with $q(a_2+) = q(a_2-) = +\infty$. If b_2 is nonzero and not large, there is an extremum near the former pole $x = a_2$ pointing away from the x-axis. As b_2 gets large, $q(x)$ flattens and tends toward the axis quickly with $q(x) = \mathcal{O}\left(b_2{}^{-2}\right)$ and $q'(a_2) = \mathcal{O}\left(b_2{}^{-2}\right)$. The extremum disappears as it crosses the horizontal asymptote $y = 1$.

When the complex roots are sufficiently close to the x-axis, these two cases can be summarized as: *complex roots in the numerator can create extrema pointing toward the x-axis and complex roots from the denominator can create extrema pointing away from the x-axis*. These cases are depicted in Figure (1).

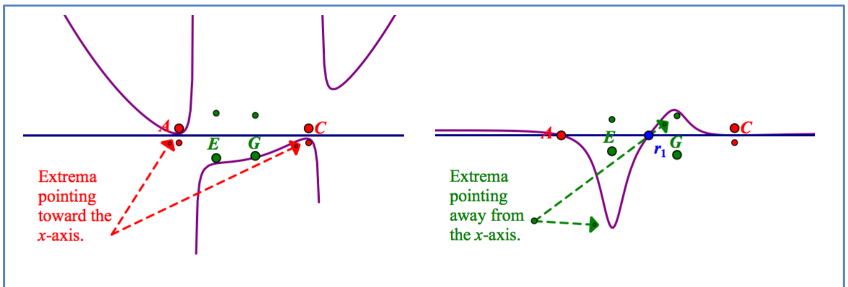

Figure (1)

4.2 Vertically Stacked Roots

When complex roots from either the numerator or the denominator are vertically stacked above (1) real root from either the numerator or the denominator or (2) other complex roots in either the numerator or the denominator, specific behavior can be recognized in the graph of the rational function. This leads to the three following theorems.

Theorem 2. Suppose $f(x)=\frac{f(x)}{j(x)}\times\frac{(x-a)^2+b^2}{(x-a)^2+c^2}$ where f/j has neither a zero nor a pole near a. Let $\frac{f(x)}{j(x)}=Q(x)$. Then

$$f(x)=Q(x)\cdot\left[1+\frac{b^2-c^2}{(x-a)^2+c^2}\right].$$

Thus, for $|x-a|<\delta$, where δ is some small, nonnegative value,

$$f(x)\approx Q(a)\cdot\left[1+\frac{b^2-c^2}{c^2}\right].$$

Therefore, for x near a, the "stacked" complex roots will only affect the function in proportion to the relative difference between their respective imaginary parts. [A small calculation proves this result. Since $x-a$ is small, $(x-a)^2+c^2$ is approximately c^2.] Figure (2) represents two versions of a function. In the left version, the imaginary values are more separated vertically. In the right version, the imaginary values are less separated vertically.

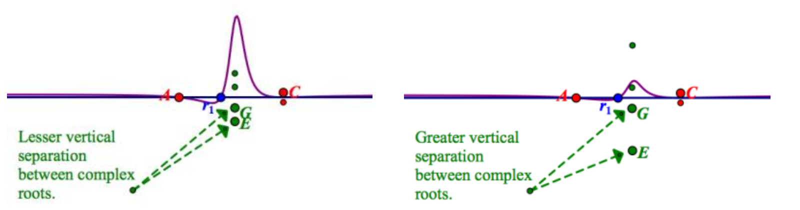

Figure (2).

Theorem 3. For both x near a and b close to c, we have

$$f(x)=Q(x)\cdot(1+\epsilon(x))\approx Q(a), \text{ where } \varepsilon(x)=\frac{b^2-c^2}{c^2}.$$

Thus,

$$f''(a)=\underbrace{\frac{b^2}{c^2}\cdot Q''(a)}_{goes\ to\ 0\ if\ a\ is\ an\ IP}+Q(a)\cdot\frac{2}{c^2}\left(1-\frac{b^2}{c^2}\right)$$

and if f has an inflection point (IP) at $x=a$, then the above shows how the inflection point is perturbed by the complex roots. That is, the point of inflection shifts one way if $b>c$, the other when $b<c$, and does not shift when $b=c$. For example, if f changes from concave down to up, then $b^2>c^2$ shifts the inflection point to the right.

Theorem 4.

$$\text{For } f(x)=\frac{(x-r_1)\left((x-a_1)^2+b_1^2\right)}{(x-r_2)\left((x-a_2)^2+b_2^2\right)}, \quad \left|f''(r_i)\right|>\left|f''\left(\underbrace{r}_{a_i\to r}\right)\right|.$$

$$\text{For } f(x)=\frac{\left((x-a_1)^2+b_1^2\right)\left((x-a_2)^2+b_2^2\right)}{\left((x-a_3)^2+b_3^2\right)\left((x-a_4)^2+b_4^2\right)}, \quad \left|f''(a_i)\right|>\left|f''\left(\underbrace{a_i}_{a_j\to a_i}\right)\right|.$$

Thus, since $|f''(x)|$ diminishes in the region of the stacked roots, the absolute value of the concavity of the function diminishes in that region. In other words, the function "flattens" in that region.

5. Graphical complex behavior

Dynamic HTML applets are provided through hyperlinks in order to investigate the following ideas. In these applets, complex roots from either the numerator (denoted by red points) or denominator (denoted by green points) can be dragged about the coordinate plane to investigate the effect of the location of the complex roots to the graph of the function. Notably, when the complex roots are dragged to lie on the x-axis (and thus become real roots – the additional real root denoted by blue points) they split into two distinct real roots which can then be moved on the x-axis. [Note that complex roots: $A \pm bi$ become real roots A and r_1; $C \pm di$ become real roots C and r_2; $E \pm fi$ become real roots E and r_3; and $G \pm hi$ become real roots G and r_4.] When the appropriate real root is dragged off from the x-axis, the other real root disappears and the dragged real root becomes a pair of complex roots. This all occurs for real and complex roots from both the numerator and the denominator.

Additionally, each graphing applet accessed through the hyperlink provides two means of investigation. First, the graphs provide suggested START and MOVE TO points to demonstrate ideas. Second, these suggested points can be hidden or shown with the press of a button on the sketch. Thus, after investigating the suggested point positions, the user can hide the suggestions and experiment further with the dynamic sketch without the clutter of additional points and textual directions.

The remainder of this investigation includes observations regarding the behavior of rational functions. In some cases, the complex and real roots are sufficiently horizontally separated so that they act as they are distinct and not overlapping Case 1. In other scenarios, some of the complex roots stack above some other real or complex roots and their distinction may not be recognized. We consider these as two cases: Case 2 contains at least one real root in both the numerator and the denominator and Case 3 contains no real roots in either the numerator or the denominator. Thus, we must consider all rational functions through three cases.

5.1 Case 1

Given that all values r_{n_j}, r_{d_k}, a_{n_j} and a_{d_k} are sufficiently separated so as to not create the behavior of overlapping or stacked roots.

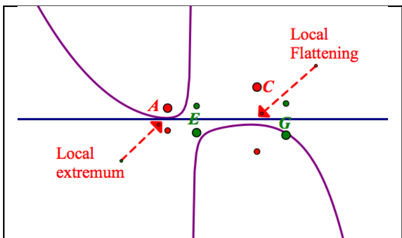

Figure (Case 1A).

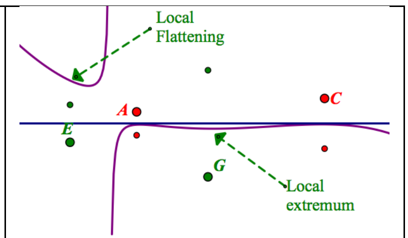

Figure (Case 1B).

Case 1A. In the region of a_{n_j}, the graph may possess either a local extremum pointing toward the *x*-axis (for small values of *b*) or a local flattening (diminishing the absolute value of the concavity) of the graph (for medium values of *b*). (See Figure (Case 1A).) To experiment, use the applet located at the URL https://php.radford.edu/~ejmt/v12n2n1/Case1A. (Note. In all dynamic applets in this paper, recommended moves of points are provided to the reader. However, to allow unfettered experimentation, all real and complex roots are movable. If readers move too many points and get confused by the resulting actions in the graph, they can simply open the applet anew and the graph will be in its original form.)

Case 1B. In the region of a_{d_k}, the graph may possess either a local extremum (for small values of *b*) pointing vertically away from a non-vertical (horizontal, oblique, or curved) asymptote or a local flattening (diminishing the absolute value of the concavity) (for medium values of *b*). (See Figure (Case 1B).) To experiment, use the applet located at the URL https://php.radford.edu/~ejmt/v12n2n1/Case1B.

5.2 Case 2

A number of the observations in the section are connected to the Rational Function Theorems previously posed. Cases for these conjectures are based on some values r_{n_j}, r_{d_k}, a_{n_j} and a_{d_k} being insufficiently separated so as to avoid the behavior of overlapping roots. However, this leads to a number of subcases. Case 2 considers when complex roots from either the numerator or denominator are vertically stacked with real roots from either the numerator or denominator.

Prior to considering these cases, a simple notion can be seen as driving the results: *real roots win*. Simple algebraic manipulation will demonstrate that the real roots of a rational function cannot be removed. Thus, real roots in the numerator will produce *x*-intercepts and real roots in the denominator will produce vertical asymptotes, regardless of the location of the complex roots. Thus, informally, we can state that the *real roots in either the numerator or denominator overpower any complex roots*. This can be demonstrated through:

For a real root in the numerator divided by a complex root in the denominator: Suppose $f(x) = \frac{\phi(x)}{\psi(x)} \cdot \frac{x-r}{(x-a)^2+b^2}$, where ϕ/φ has neither zero nor pole near a, with $a \approx r$ and $x \approx a$. Then

$$f(x) \approx Q(x) \cdot \frac{0}{0^2+b^2} \approx Q(a) \cdot 0 = 0.$$

Informally, we can summarize this to mean that the real root in the numerator wins. The complex root can have no effect on the graph intersecting the *x*-axis at the real zero.

For a real complex root in the numerator divided by a real root in the denominator: Suppose $f(x) = \frac{\phi(x)}{\psi(x)} \cdot \frac{(x-a)^2+b^2}{x-r}$, where ϕ/φ has neither zero nor pole near a, with $a \approx r$ and $x \approx a$. Then

$$f(x) = Q(x) \cdot \left(\frac{(x-a)^2}{x-r} + \frac{b^2}{x-r}\right) \approx Q(a) \cdot (\pm\infty) = sign(Q(a)) \cdot \infty.$$

Informally, we can summarize this to mean that the real root in the denominator wins. The complex root can have no effect on the graph possessing a pole at the real zero in the denominator.

However, the complex roots still have some effect on the graph. Notably, the complex roots cannot produce any additional *x*-intercepts or vertical asymptotes. Thus, the only other possible effect that complex roots vertically stacked has with real roots at r_i is to diminish the value of $|f''(r_i)|$.

Case 2A, $a_{n_j} \sim r_{n_j}$: complex roots in the numerator are stacked with a real root in the numerator. The graph may possess a flattening near the real root r_{n_j}. (See Figure (Case 2A).) To experiment, use this applet.

Case 2B, $a_{n_j} \sim r_{d_k}$: complex roots in the numerator are stacked with a real root in the denominator. The graph may possess a flattening near the real pole at r_{d_k}. (See Figure (Case 2B).) To experiment, use this applet.

Case 2C, $a_{d_k} \sim r_{n_j}$: complex roots in the denominator are stacked with a real root in the numerator. The graph may possess a flattening near the real root r_{n_j}. (See Figure (Case 2C).) To experiment, use this applet.

Case 2D, $a_{d_k} \sim r_{d_k}$: complex roots in the denominator are stacked with a real root in the denominator. The graph may possess a flattening near the real pole at r_{d_k}. (See Figure (Case 2D).) To experiment, use this applet.

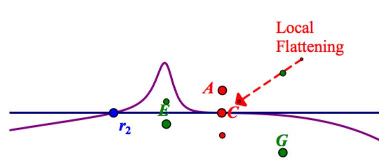

Figure (Case 2A).

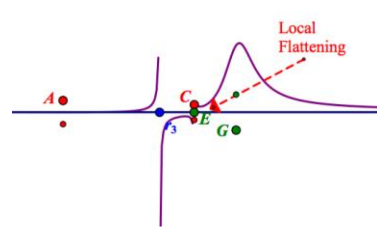

Figure (Case 2B).

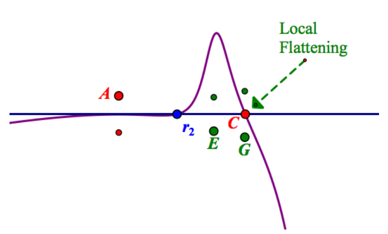

Figure (Case 2C).

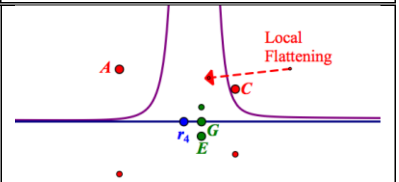

Figure (Case 2D).

5.3 Case 3

Case 3A, $a_{n_1} \sim a_{n_2}$: two pairs of complex roots in the numerator are stacked. The graph may possess a flattened local extremum pointing toward the *x*-axis. (See Figure (Case 3A).) To experiment, use this applet.

Case 3B, $a_{d_1} \sim a_{d_2}$: two pairs of complex roots in the denominator are stacked. The graph may possess a flattened local extremum pointing vertically away from a non-vertical asymptote (horizontal, oblique, or curved) or a local flattening. (See Figure (Case 3B).) To experiment, use this applet.

Case 3C, $a_{n_j} \sim a_{d_k}$: a pair of complex roots from the numerator are stacked with a pair of complex roots from the denominator.

Since $a_{n_j} \approx a_{d_k}$, for a moment we will treat them as equal. Then, as seen in Theorem 3,

$$\frac{\left(x-a_{n_j}\right)^2+b_{n_j}^2}{\left(x-a_{d_k}\right)^2+b_{d_k}^2}=\frac{(x-a)^2+b_{n_j}^2}{(x-a)^2+b_{d_k}^2}=1-\frac{b_{n_j}^2-b_{d_j}^2}{(x-a)^2+b_{d_k}^2}.$$

This reveals that the numerator becomes a constant and that the complex roots remain in the denominator. Thus, informally, we can state that the *complex roots in the denominator overpower the complex roots in the numerator*. Thus, the graph may possess either a local extremum (for small values of *b*) pointing away from a non-vertical asymptote (horizontal, oblique, or curved) or a local flattening (for medium values of *b*). (See Figure (Case 3C).) To experiment, use this applet.

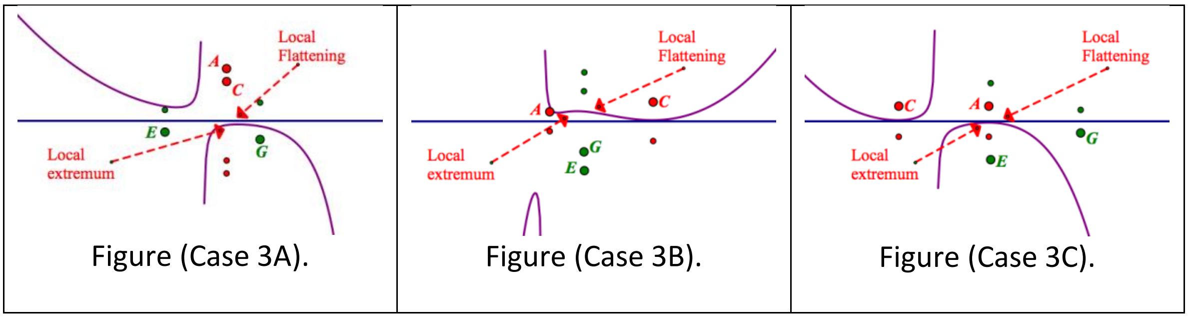

Figure (Case 3A). Figure (Case 3B). Figure (Case 3C).

6. Conclusions, Comments, and Invitations

Having previously determined the location of complex roots based on the graphs of quadratic, cubic, quartic, and quintic real monic polynomials, this investigation now concludes with the location of complex roots from either the numerator or denominator of rational functions. In summary, when particular conditions are met regarding the horizontal stacking or separation of real and complex roots and the proximity of the complex roots to the *x*-axis, approximate locations of complex roots can be determined.

The authors are particularly happy that this series of investigation has culminated in these findings. This is particularly the case when we remember that the seed question for this series of investigations initiated more than a quarter century ago. While this has led to fruitful results and publications, we feel the need to move on to other types of investigations.

We invite the reader to look further into these ideas and make additional discoveries. We invite them to dare to imagine, investigate, conjecture, and, hopefully, make findings far beyond ours. In conclusion, we simply state, “Tag. You’re it.”

7. References

1. Bailey, D. H., and Borwein, J. M., 2005: Experimental mathematics: Examples, methods and implications. *Notices of the AMS*, **52(5)**, 502-514.
2. Bauldry, W., Bossé, M. J., and Otey, S. H., 2017a, in press: Circle constructions of complex polynomial roots. Electronic Journal of Mathematics and Technology (eJMT).
3. Bossé, M. J., and Bauldry, W., 2017b, in press: Locating complex roots on quintic functions. *The Mathematics Enthusiast*.
4. Bauldry, W., Bossé, M. J., and Otey, S. H., 2017c, in press: Visualizing complex roots. *The Mathematics Enthusiast*.
5. Ellis, W., Bauldry, W., Bossé, M. J., and Otey, S. H., 2016; Employing technology to visualize complex roots of real polynomials. *Proceedings of Technology and its Integration in Mathematics Education*, Mexico City, June 29 - July 2, 2016.